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Quasi-measures and dimension theory

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Abstract

The theory of quasi-measures on compact Hausdorff spaces has been initiated by J. Aarnes. Here the relation of quasi-measures to classical notions of dimension theory is studied in the more general setting of normal spaces. It is shown that if X is normal and the large inductive dimension $\operatorname{Ind}(X) \leqslant 1$, then every quasi-measure on X admits a unique extension to a closed regular, finitely additive measure on the Borel algebra of X. As a consequence, every quasi-linear functional on $C_b(X)$, the space of bounded continuous real valued functions on X, is linear.

Keywords: Borel set; Continuous; Dimension; Linear functional; Measure; Normal

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1. Introduction

The purpose of this paper is to demonstrate that the theory of quasi-measures introduced by Aarnes [1–5] is closely tied to several concepts from the classical dimension theory of normal topological spaces: the large inductive dimension $\operatorname{Ind}(X)$, the Lebesgue covering dimension $\dim(X)$, and the analytic dimension $\operatorname{ad}(X)$ of the Banach algebra of bounded continuous real valued functions on X. The reader is referred to Pears [13] for a detailed discussion of these concepts and their interrelationships. The small inductive dimension $\operatorname{ind}(X)$ does not appear to have comparable significance in the study of quasi-measures.

The work of Aarnes yields striking examples of quasi-measures on the unit square in the plane and the two-dimensional sphere which are not finitely subadditive [3,4]. These results are systematized and extended in [5] and also in the work of Knudsen [12]. Here it is shown that for normal spaces X with $\operatorname{Ind}(X) \leqslant 1$ (respectively, $\dim(X) \leqslant 1$), every quasi-measure (respectively, $\{0,1\}$-valued quasi-measure) is subadditive, and admits a finitely additive extension to the smallest algebra of subsets of X which contains the closed sets. Thus the examples given by Aarnes are in some sense minimal. To complicate

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the picture slightly, we give an example of a compact Hausdorff space S with $\dim(S) = 1$, $\operatorname{Ind}(S) = 2$, for which every quasi-measure is subadditive.

The paper is organized as follows: Section 2 summarizes basic definitions and results for quasi-measures and quasi-linear functionals, and establishes a framework for the discussion in the setting of normal spaces. Section 3 points out the equivalence of subadditivity of quasi-measures with other desirable properties. Section 4 treats several sufficient conditions for subadditivity which are far too strong to be necessary. Section 5 discusses $\{0, 1\}$-valued quasi-measures and multiplicative quasi-linear functionals. Section 6 gives the major result of the paper, subadditivity of quasi-measures on normal spaces X with $\operatorname{Ind}(X) \leqslant 1$, and Section 7 is devoted to the example mentioned above with $\dim(S) = 1$, $\operatorname{Ind}(S) = 2$. Section 8 indicates how certain extensions of the results for normal spaces to the larger setting of completely regular Hausdorff spaces may be achieved.

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2. Basic definitions and results

Any needed information about general topology and dimension theory can be found in [10,13]. If A is a subset of a topological space X, then the complement, boundary, closure, and interior of A are denoted by $\widetilde{A}$, $\partial(A)$, $\mathrm{cl}(A)$, and $\mathrm{int}(A)$ (or, for example, $\mathrm{int}_X(A)$, if confusion might arise). In this paper, "normal" is taken to imply the Hausdorff property. A space is totally disconnected if its connected components are points. If $\dim(X) = 0$ (equivalently, if $\operatorname{Ind}(X) = 0$), then X is totally disconnected; the converse holds if X is a compact Hausdorff space.

If X is a completely regular Hausdorff space, then $C_b(X)$ denotes the space of bounded continuous real valued functions on X. Under the norm $\|f\| = \sup\{|f(x)|\colon x \in X\}$, this space is a commutative C^*-algebra with identity. As usual, βX denotes the Stone–Čech compactification of X; $C_b(X)$ is isomorphic (as a C^*-algebra) to $C(\beta X)$ under the map which sends $f \in C_b(X)$ to its unique continuous extension f^β.

If $E \subseteq X$, then χ_E denotes the characteristic function of E. If $E \subseteq X$ and $f \in C_b(X)$, we write $E \prec f$ if $0 \leqslant f \leqslant 1$ and $f|E \equiv 1$.

A subset Z of X is a zero set of X if there is an $f \in C_b(X)$ such that $Z = f^{-1}(0)$. Complements of zero sets are called cozero sets. The symbol $\mathcal{Z}$ (respectively, $\mathcal{F}$) denotes the family of zero sets (respectively, closed sets) of X. The smallest algebra of subsets of X containing $\mathcal{Z}$ (respectively, $\mathcal{F}$) is called the Baire algebra $Ba^*(X)$ (respectively, the Borel algebra $Bo^*(X)$). The corresponding σ-algebras of Baire and Borel sets are denoted by $Ba(X)$ and $Bo(X)$.

The classical theory of measures on topological spaces is discussed in [14]. The theory of quasi-measures introduced in [3,4] for compact Hausdorff spaces has been extended to the setting of completely regular Hausdorff spaces by J. Boardman [8].

Definition 2.1. A Baire (respectively, Borel) *quasi-measure* on a completely regular Hausdorff space X is a function $\mu : \mathcal{Z}$ (respectively, $\mathcal{F}$) $\to \mathbb{R}^+$ such that for all $A_1, A_2 \in \mathcal{Z}$ (respectively, $\mathcal{F}$):

(i) $A_1 \subseteq A_2 \Rightarrow \mu(A_1) \leqslant \mu(A_2)$,

(ii) $A_1 \cap A_2 = \emptyset \Rightarrow \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$,

(iii) given A_1 and $\varepsilon > 0$, $\exists A_2$ such that $A_1 \cap A_2 = \emptyset$ and $\mu(A_1) + \mu(A_2) > \mu(X) - \varepsilon$.

This formulation is entirely equivalent to that given in [3,8], where, for example, a Borel quasi-measure is defined originally on any set which is either open or closed. If μ satisfies (i)–(iii) above for closed sets, then for any open set G, the stipulation that $\mu(G) = \mu(X) - \mu(X \setminus G)$ recovers the Aarnes formulation, with (iii) yielding the regularity condition $\mu(G) = \sup\{\mu(F)\colon F \text{ closed}, F \subseteq G\}$. Moreover, a strengthened version of (ii) then holds: if A_0 is the disjoint union of sets $A_1, A_2, \ldots, A_n$, and each A_i is either open or closed, $0 \leqslant i \leqslant n$, then

$$\mu(A_0) = \sum_{i=1}^{n} \mu(A_i)$$

[3, Proposition 2.1(c)]. The analogous statements hold for Baire quasi-measures [8].

The primary difficulty with a quasi-measure is that it need not be finitely subadditive. Thus the standard processes of measure theory can only be performed on pairwise disjoint families of sets, unless one has additional information about the quasi-measure (cf. Proposition 3.1 below).

Definition 2.2. If X is a completely regular Hausdorff space, then a functional $\rho : C_b(X) \to \mathbb{R}$ is said to be *quasi-linear* if:

(i) $\rho(f) \geqslant 0$ whenever $f \geqslant 0$; and

(ii) for each $h \in C_b(X)$, the restriction of ρ to $A(h)$, the supremum norm closed subalgebra of $C_b(X)$ generated by h and 1, is linear.

Information about quasi-linear functionals on general C^*-algebras can be found in [1,2]. There is a natural one-to-one correspondence between quasi-linear functionals on $C_b(X)$ and $C(\beta X)$, given by $\rho(f) = \rho^\beta(f^\beta)$. If $\rho(1) = 1$, then ρ is called a quasi-state of $C_b(X)$. If ρ is a quasi-linear functional on $C_b(X)$, then for all $f, g \in C_b(X)$:

(i) $|\rho(f) - \rho(g)| \leqslant \rho(1) \cdot \|f - g\|$,

(ii) if $f \leqslant g$, then $\rho(f) \leqslant \rho(g)$,

(iii) $\rho(c \cdot f) = c \cdot \rho(f)$ for all $c \in \mathbb{R}$,

(iv) $\rho(f + g) = \rho(f) + \rho(g)$ whenever the pointwise product $f \cdot g = 0$.

The primary difficulty with a quasi-linear functional is that it need not be additive, nor even subadditive: $\rho(f + g) > \rho(f) + \rho(g)$ may occur for nonnegative $f, g \in C_b(X)$. An explicit example of this phenomenon appears in [3, Section 7].

In [3], Aarnes extended the Riesz Representation Theorem by extablishing a one-to-one correspondence between Borel quasi-measures on a compact Hausdorff space K and quasi-linear functionals on $C(K)$. In [8], Boardman modified these ideas to obtain a one-to-one correspondence between Baire quasi-measures on a completely regular Hausdorff space X and quasi-linear functionals on $C_b(X)$, thus extending the Alexandroff Representation Theorem [14, Theorem 5.1]. The main problem in the completely regular setting is that the elegant spectral theory and functional calculus presented by Aarnes in the compact case must apparently be relinquished, while retaining the integral representation of ρ in modified form.

The correspondence goes as follows: if $\rho: C_b(X) \to \mathbb{R}$ is quasi-linear and $Z \in \mathcal{Z}$, then $\mu(Z) = \inf\{\rho(f)\colon f \in C_b(X),\ Z \prec f\}$. Conversely, if μ is a Baire quasi-measure on X with $\mu(X) = 1$, and $f \in C_b(X)$, define $\hat{f}: \mathbb{R} \to \mathbb{R}$ by $\hat{f}(t) = \mu(\{x \in X\colon f(x) \geqslant t\})$. Then $\hat{f}$ is a monotone decreasing function, and if λ_1, λ_2 are any real numbers such that range$(f) \subseteq [\lambda_1, \lambda_2]$, then

$$\rho(f) = \lambda_1 + \int_{\lambda_1}^{\lambda_2} \hat{f}(t)\,\mathrm{d}t.$$

Since ρ corresponds to $\rho^\beta: C(\beta X) \to \mathbb{R}$, it also induces a Borel quasi-measure ν^β on βX, via [3]. If W is a zero set of βX, then

$$\begin{aligned}\nu^\beta(W) &= \inf\{\rho^\beta(g)\colon g \in C(\beta X),\ W \prec g\} \\ &\geqslant \inf\{\rho(h)\colon h \in C_b(X),\ W \cap X \prec h\} = \mu(W \cap X).\end{aligned}$$

In general, the inequality is strict: consider, for example, $X = \mathbb{N}$, $W = \beta\mathbb{N} \setminus \mathbb{N}$, and $\rho(f) = f^\beta(p)$, where p is a fixed member of $\beta\mathbb{N} \setminus \mathbb{N}$.

We now show that for normal spaces, the notions of Baire and Borel quasi-measures are entirely equivalent.

Proposition 2.3. *Let X be a normal space.*

(a) *If $\nu: \mathcal{F} \to \mathbb{R}^+$ is a Borel quasi-measure on X, then $\mu = \nu|\mathcal{Z}$ is a Baire quasi-measure.*

(b) *If $\mu: \mathcal{Z} \to \mathbb{R}^+$ is a Baire quasi-measure on X, then $\nu(F) = \inf\{\mu(Z)\colon F \subseteq Z\}$ is a Borel quasi-measure on X such that $\nu|\mathcal{Z} = \mu$. Moreover, ν is the unique extension of μ to a Borel quasi-measure on X.*

Proof. (a) For Definition 2.1(iii), let $Z_1 \in \mathcal{Z}$ and $\varepsilon > 0$. There is a closed set F with $Z_1 \cap F = \emptyset$ and $\nu(Z_1) + \nu(F) > \nu(X) - \varepsilon$. Since X is normal, there is a zero set Z_2 with $F \subseteq Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Then $\mu(Z_1) + \mu(Z_2) = \nu(Z_1) + \nu(Z_2) \geqslant \nu(Z_1) + \nu(F) > \nu(X) - \varepsilon = \mu(X) - \varepsilon$.

(b) Property 2.1(i) is immediate. For (ii), let $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \emptyset$, and let $\varepsilon > 0$. Since X is normal, there are zero sets Z_1, Z_2 with $Z_1 \cap Z_2 = \emptyset, Z_i \supseteq F_i$, and $\mu(Z_i) < \nu(F_i) + \varepsilon/2$, $i = 1, 2$. Then $\nu(F_1 \cup F_2) \leqslant \mu(Z_1 \cup Z_2) = \mu(Z_1) + \mu(Z_2) < \nu(F_1) + \nu(F_2) + \varepsilon$, and so $\nu(F_1 \cup F_2) \leqslant \nu(F_1) + \nu(F_2)$. Conversely, let Z

be a zero set containing $F_1 \cup F_2$ with $\mu(Z) < \nu(F_1 \cup F_2) + \varepsilon$. Since X is normal, there are zero sets Z_1, Z_2 with $Z_1 \cup Z_2 \subseteq Z$, $Z_1 \cap Z_2 = \emptyset$, $F_1 \subseteq Z_1$, $F_2 \subseteq Z_2$. Then $\nu(F_1) + \nu(F_2) \leqslant \mu(Z_1) + \mu(Z_2) = \mu(Z_1 \cup Z_2) \leqslant \mu(Z) < \nu(F_1 \cup F_2) + \varepsilon$. Hence $\nu(F_1) + \nu(F_2) \leqslant \nu(F_1 \cup F_2)$.

For (iii), let $F_1 \in \mathcal{F}$ and $\varepsilon > 0$. Choose $Z_1 \supseteq F_1$ with $\mu(Z_1) < \nu(F_1) + \varepsilon/2$. Choose Z_2 such that $Z_1 \cap Z_2 = \emptyset$ and $\mu(Z_1) + \mu(Z_2) > \mu(X) - \varepsilon/2$. Then $\nu(F_1) + \nu(Z_2) = \nu(F_1) + \mu(Z_2) > \mu(Z_1) + \mu(Z_2) - \varepsilon/2 > \mu(X) - \varepsilon = \nu(X) - \varepsilon$, as desired.

For uniqueness, suppose λ is a Borel quasi-measure on X such that $\lambda|\mathcal{Z} = \mu$. Then $\lambda(F) \leqslant \inf\{\lambda(Z)\colon F \subseteq Z\} = \inf\{\mu(Z)\colon F \subseteq Z\} = \nu(F)$ for all $F \in \mathcal{F}$. Suppose $\nu(F_1) - \lambda(F_1) = \delta > 0$ for some F_1. Then for any $F_2 \subseteq X \setminus F_1$, $\lambda(F_1) + \lambda(F_2) \leqslant \nu(F_1) - \delta + \nu(F_2) \leqslant \nu(X) - \delta = \mu(X) - \delta = \lambda(X) - \delta$, and so (iii) fails for λ, a contradiction. Hence $\lambda = \nu$. $\square$

Note that from this result, μ is $\{0,1\}$-valued if and only if ν is $\{0,1\}$-valued. The category of normal spaces X has another advantage from our point of view: $\dim(X) = \dim(\beta X)$ and $\operatorname{Ind}(X) = \operatorname{Ind}(\beta X)$ [13, pp. 232–233]. Consequently, in Sections 3–7 of this paper we shall assume (unless otherwise noted) that *all spaces X are normal, and any quasi-measure is defined on all closed subsets of X*. In Section 8, we indicate those results which remain true for Baire quasi-measures on completely regular spaces.

3. Finite subadditivity of quasi-measures

We begin with an essential link between quasi-measures and finitely additive Borel measures.

Proposition 3.1. *Let ν be a Borel quasi-measure on a normal space X. Then the following are equivalent:*

(a) *If F_1, F_2 are closed subsets of X, then $\nu(F_1 \cup F_2) \leqslant \nu(F_1) + \nu(F_2)$.*

(b) *If U_1, U_2 are open subsets of X, then $\nu(U_1 \cup U_2) \leqslant \nu(U_1) + \nu(U_2)$.*

(c) *ν admits a (unique) extension to a finitely additive, closed regular measure λ on the Borel algebra $Bo^*(X)$.*

Proof. Note that in (b), $\nu(U)$ is defined to be $\nu(X) - \nu(X \setminus U)$. In (c), λ is said to be closed regular if $\lambda(B) = \sup\{\lambda(F)\colon F \text{ closed}, F \subseteq B\}$ for each $B \in Bo^*(X)$. This yields the uniqueness of the extension. If X is compact, it is well known that (c) can be replaced by the stronger condition (c′): ν admits a unique extension to a countably additive, closed regular measure λ on the Borel σ-algebra $Bo(X)$.

(c) $\Rightarrow$ (a): $\nu(F_1 \cup F_2) = \lambda(F_1 \cup F_2) = \lambda(F_1) + \lambda(F_2 \setminus F_1) \leqslant \lambda(F_1) + \lambda(F_2) = \nu(F_1) + \nu(F_2)$. Note that $F_2 \setminus F_1$ is neither open nor closed in general, so that $\nu(F_2 \setminus F_1)$ need not be defined.

(a) $\Rightarrow$ (b): Let U_1, U_2 be open in X, and let F be a closed subset of $U_1 \cup U_2$. Then $F \setminus U_1$ and $F \setminus U_2$ are disjoint closed sets, so there are disjoint open sets W_1, W_2 with $F \setminus U_i \subseteq W_i$, $i = 1, 2$ (this is the only point in the proof where normality is used). Let $A_i = F \setminus W_i$, $i = 1,2$. Then A_1, A_2 are closed, $A_1 \cup A_2 = F$, and $A_i \subseteq U_i$, $i = 1,2$. Now $\nu(F) = \nu(A_1 \cup A_2) \leqslant \nu(A_1) + \nu(A_2) \leqslant \nu(U_1) + \nu(U_2)$. Hence $\nu(U_1 \cup U_2) = \sup\{\nu(F)\colon\ F \subseteq U_1 \cup U_2\} \leqslant \nu(U_1) + \nu(U_2)$, as desired.

(b) $\Rightarrow$ (c): This is very similar to the proof in [11] that any regular content on a locally compact Hausdorff space extends to a compact regular Borel measure, so we only sketch the details. First, for any subset E of X, define $\nu^*(E) = \inf\{\nu(G)\colon\ G \text{ open}, E \subseteq G\}$. Then $\nu^*(\emptyset) = 0$, ν^* is monotone, and ν^* is finitely subadditive. Define a subset E of X to be ν^*-measurable if $\nu^*(D) = \nu^*(D \cap E) + \nu^*(D \cap \widetilde{E})$ for every subset D of X. Proceeding as in [11, p. 234, Theorem D], a subset E of X is ν^*-measurable if and only if $\nu(G) = \nu^*(G) \geqslant \nu^*(G \cap E) + \nu^*(G \cap \widetilde{E})$ for each open set G.

Now an argument as in [11, pp. 234–235, Theorem E] shows that each closed subset of X is measurable, and an argument similar to [11, pp. 44–45, Theorem A] shows that the family $\mathcal{A}$ of ν^*-measurable sets is an algebra of subsets of X. Hence $\mathcal{A} \supseteq Bo^*(X)$.

Let $\lambda = \nu^*|Bo^*(X)$. If $E_1, E_2 \in Bo^*(X)$ with $E_1 \cap E_2 = \emptyset$, then $\lambda(E_1 \cup E_2) = \nu^*(E_1 \cup E_2) = \nu^*((E_1 \cup E_2) \cap E_1) + \nu^*((E_1 \cup E_2) \cap \widetilde{E}_1) = \nu^*(E_1) + \nu^*(E_2) = \lambda(E_1) + \lambda(E_2)$, so λ is finitely additive. Finally, if $E \in Bo^*(X)$, then $\lambda(\widetilde{E}) = \nu^*(\widetilde{E}) = \inf\{\nu(G)\colon\ G \text{ open}, \widetilde{E} \subseteq G\}$, so $\lambda(E) = \sup\{\nu(F)\colon\ F \text{ closed}, F \subseteq E\}$. Hence λ is closed regular. $\square$

If the extension of ν to λ described in (c) is going to fail, it will do so at the very first level of complexity beyond closed sets and open sets: namely, sets of the form $F \cap G$, where F is closed and G is open. This situation can be seen concretely in the fundamental example given by Aarnes on the unit square S [3, Section 6]. Indeed $E = \{(x,0)\colon\ 0 < x < 1\}$ has the form $F \cap G$. Now E can be represented as the bottom edge of S, minus two points, yielding $\lambda(E) = 0 - 0 = 0$. But also E is the entire boundary of the square, minus the union of the left, right, and top sides, yielding $\lambda(E) = 1 - 0 = 1$. Hence no consistent extension to E is possible. In general, if ν can be consistently extended to sets of the form $F_2 \setminus F_1 = F_2 \cap (X \setminus F_1)$, with preservation of monotonicity and finite additivity, then an argument as in (c)$\Rightarrow$(a) above shows that ν is subadditive on closed sets and therefore extends to the entire Borel algebra.

Corollary 3.2. *Let ν be a Borel quasi-measure on a normal space X, and assume that $\nu(D) = 0$ for every closed nowhere dense set D. Then ν extends (uniquely) to a finitely additive, closed regular measure on $Bo^*(X)$.*

Proof. Let F_1, F_2 be closed subsets of X, and let $U_1 = \operatorname{int}(F_1)$, $U_2 = \operatorname{int}(F_2 \setminus F_1)$, $T = (F_1 \cup F_2) \setminus (U_1 \cup U_2)$. Then U_i is an open subset of F_i, $i = 1,2$, and T is closed and nowhere dense. To see the last assertion, suppose that W is a nonempty open subset of T, and let $x_0 \in W$. Then for any open set G with $x_0 \in G \subseteq W$, $G \not\subseteq F_1$ (since $x_0 \notin U_1$), and $G \not\subseteq F_2 \setminus F_1$ (since $x_0 \notin U_2$). Thus every neighborhood of x_0 meets both $F_2 \setminus F_1$ and F_1, so $x_0 \in \partial(F_1)$, and $W \subseteq \partial(F_1)$. Since the boundary of any closed set is nowhere dense, we have a contradiction.

Now $F_1 \cup F_2$ is the disjoint union of U_1, U_2, and T, so $\nu(F_1 \cup F_2) = \nu(U_1) + \nu(U_2) + \nu(T) \leqslant \nu(F_1) + \nu(F_2)$, and Proposition 3.1 can be applied. □

Note that in the example of Aarnes mentioned above, the boundary of the unit square in the plane is a closed nowhere dense set with full "measure".

Theorem 3.3. *Let X be a normal space. Then there is a one-to-one correspondence between quasi-linear functionals ρ on $C_b(X)$ and Borel quasi-measures ν on X. Moreover, the following are equivalent:*

(a) *ρ is linear (hence a positive linear functional on $C_b(X)$).*

(b) *If $f, g \geqslant 0$ in $C_b(X)$, then $\rho(f+g) \leqslant \rho(f) + \rho(g)$.*

(c) *If $F_1, F_2 \in \mathcal{F}$, then $\nu(F_1 \cup F_2) \leqslant \nu(F_1) + \nu(F_2)$.*

(d) *ν extends (uniquely) to a finitely additive, closed regular measure λ on $Bo^*(X)$.*

Proof. The correspondence between ρ and ν follows from [8] and 2.3. Clearly (a) $\Rightarrow$ (b), and (c) $\Leftrightarrow$ (d) by Proposition 3.1.

(b) $\Rightarrow$ (c): Let $\varepsilon > 0$, and choose zero sets Z_1, Z_2 with $Z_i \supseteq F_i$ and $\nu(Z_i) < \nu(F_i) + \varepsilon/4$, $i = 1, 2$ (cf. Proposition 2.3). Now choose $f_1, f_2 \in C_b(X)$ with $Z_i \prec f_i$ and $\rho(f_i) < \nu(Z_i) + \varepsilon/4$, $i = 1, 2$. Let $g = \min(f_1 + f_2, 1)$. Then $Z_1 \cup Z_2 \prec g$, and so $\nu(F_1 \cup F_2) \leqslant \nu(Z_1 \cup Z_2) \leqslant \rho(g) \leqslant \rho(f_1 + f_2) \leqslant \rho(f_1) + \rho(f_2) < \nu(F_1) + \nu(F_2) + \varepsilon$. The result follows.

(d) $\Rightarrow$ (a): The results of [8] show that ρ is linear if and only if $\mu = \nu|\mathcal{Z}$ extends to a finitely additive, zero set regular measure on the Baire algebra $Ba^*(X)$. Since $\gamma = \lambda|Ba^*(X)$ is finitely additive and extends μ, it suffices to show that $\mathcal{A} = \{B \in Ba^*(X)\colon \gamma(B) = \sup\{\gamma(Z)\colon Z \subseteq B\}$ and $\gamma(\widetilde{B}) = \sup\{\gamma(Z)\colon Z \subseteq \widetilde{B}\}\}$ is an algebra of sets which contains $\mathcal{Z}$. It is routine to see that $\mathcal{A}$ is an algebra (note that λ is subadditive). If $Z_1 \in \mathcal{Z}$, let $U = X \setminus Z_1$, and let $\varepsilon > 0$. Choose a closed subset F of U with $\lambda(F) > \lambda(U) - \varepsilon = \gamma(U) - \varepsilon$. Since X is normal, there is a zero set Z_2 with $F \subseteq Z_2 \subseteq U$, and so $\gamma(Z_2) > \gamma(U) - \varepsilon$. □

In view of the isomorphism between $C_b(X)$ and $C(\beta X)$, the conditions stated in Theorem 3.3 are also equivalent to the analogous conditions on ρ^β and ν^β.

We close this section with a result which will be needed in Section 6.

Proposition 3.4. *Let μ and λ be Borel quasi-measures on a normal space X, such that $0 \leqslant \lambda(F) \leqslant \mu(F)$ for every $F \in \mathcal{F}$. Then $\nu = \mu - \lambda$ is a Borel quasi-measure on X.*

Proof. Let ρ_λ, ρ_μ be the quasi-linear functionals corresponding to λ, μ. Then $0 \leqslant \rho_\lambda \leqslant \rho_\mu$, and $\rho_0 = \rho_\mu - \rho_\lambda$ is easily seen to be a quasi-linear functional on $C_b(X)$. Let ν be the quasi-measure corresponding to ρ_0. Then we need to show that $\nu = \mu - \lambda$, or that $\mu = \lambda + \nu$.

Let Z be a zero set of X. Then $\mu(Z) = \inf\{\rho_\mu(g)\colon Z \prec g\} = \inf\{\rho_\lambda(g) + \rho_0(g)\colon Z \prec g\} \geqslant \inf\{\rho_\lambda(g)\colon Z \prec g\} + \inf\{\rho_0(h)\colon Z \prec h\} = \lambda(Z) + \nu(Z)$.

Conversely, let $\varepsilon > 0$, and choose $g \succ Z$, $h \succ Z$ with $\rho_\lambda(g) < \lambda(Z) + \varepsilon/2$, $\rho_0(h) < \nu(Z) + \varepsilon/2$. Let $\varphi = g \wedge h \succ Z$. Then $\lambda(Z) + \nu(Z) > \rho_\lambda(g) - \varepsilon/2 + \rho_0(h) - \varepsilon/2 \geqslant \rho_\lambda(\varphi) + \rho_0(\varphi) - \varepsilon = \rho_\mu(\varphi) - \varepsilon \geqslant \mu(Z) - \varepsilon$, and so $\lambda(Z) + \nu(Z) \geqslant \mu(Z)$. □

4. Linearity of quasi-linear functionals

Definition 4.1. A normal space X is an *A-space* if every quasi-linear functional on $C_b(X)$ is linear.

The correspondence $\rho(f) = \rho^\beta(f^\beta)$ shows that X is an A-space if and only if βX is an A-space. The class of A-spaces includes compact totally disconnected spaces [6], closed bounded intervals in $\mathbb{R}$, and the unit circle S^1, but not $[0,1] \times [0,1]$ and S^2 (see [3,4]). All of the positive outcomes mentioned here are subsumed under Theorem 6.1.

Proposition 4.2. *If X is a normal A-space, then so is every closed subspace Y.*

Proof. Let $\varphi: C_b(Y) \to \mathbb{R}$ be quasi-linear. Then $\rho: C_b(X) \to \mathbb{R}$ defined by $\rho(f) = \varphi(f|Y)$ is easily seen to be quasi-linear and hence linear. Now if $f, g \in C_b(Y)$, with bounded continuous extensions to X, f_1 and g_1, then $\varphi(f+g) = \rho(f_1 + g_1) = \rho(f_1) + \rho(g_1) = \varphi(f) + \varphi(g)$. □

Thus $\mathbb{R}^n$ is not an A-space for $n \geqslant 2$. As we will see in Section 6, $\mathbb{R}$ is an A-space, hence so are $\beta\mathbb{R}$ and $\beta\mathbb{R} \setminus \mathbb{R}$ (by Proposition 4.2). Thus the quotient C^*-algebra $C_b(\mathbb{R})/C_0(\mathbb{R})$ has the property that every quasi-linear functional is linear, since it is isomorphic (as a C^*-algebra) to $C(\beta\mathbb{R} \setminus \mathbb{R})$.

We turn now to general sufficient conditions for a normal space X to be an A-space. By Definition 2.2, the simplest way for this to occur is if $C_b(X)$ is itself singly generated (i.e., equal to $A(h)$ for some fixed h). This occurs if and only if X is homeomorphic to a compact subset of $\mathbb{R}$. The sufficiency is clear (let $h(x) = x$, and apply the Stone–Weierstrass Theorem). Conversely, if $C_b(X) = A(h)$, then $C(\beta X) = A(h^\beta)$, and so h^β must be one-to-one, hence a homeomorphism of βX onto a (compact) subset of $\mathbb{R}$. It follows from [10, 9.6] that $X = \beta X$.

Definition 4.3. Let X be a normal space. The *continuous rank of* X, $\operatorname{cr}(X)$, is the least cardinal number τ such that if $\{f_1, \ldots, f_n\}$ is any finite subset of $C_b(X)$, then there is a subset H of $C_b(X)$, $\operatorname{card}(H) \leqslant \tau$, such that $\{f_1, \ldots, f_n\} \subseteq A(H)$, the smallest norm-closed subalgebra of $C_b(X)$ containing H (and 1).

It is easy to see that $\operatorname{cr}(X) = \operatorname{cr}(\beta X)$, and that $\operatorname{cr}(F) \leqslant \operatorname{cr}(X)$ if F is closed in X. We have $\operatorname{cr}(X) \geqslant 1$ (unless $\operatorname{card}(X) = 1$), and the condition $\operatorname{cr}(X) \leqslant 1$ is equivalent (by an easy induction) to the requirement that if $f, g \in C_b(X)$, then there is an $h \in C_b(X)$ such that $f, g \in A(h)$. It is evident from Definitions 2.2 and 4.1 that if $\operatorname{cr}(X) \leqslant 1$, then X

is an A-space. If X is compact and totally disconnected, then $\mathrm{cr}(X) \leqslant 1$ [6]. However, the A-space S^1 satisfies $\mathrm{cr}(S^1) = 2$, as the following more general result shows.

Proposition 4.4. *Let X be a compact subset of $\mathbb{R}^n$. Then $\mathrm{cr}(X) \leqslant n$, but $\mathrm{cr}(X) \leqslant 1$ if and only if X is homeomorphic to a subset of $\mathbb{R}$.*

Proof. Since $C(X) = A(H)$, where $H = \{f_1, \ldots, f_n\}$ is the set of coordinate functions on $\mathbb{R}^n$, restricted to X, the first assertion is clear. If $\mathrm{cr}(X) \leqslant 1$, choose $h \in C(X)$ such that $\{f_1, \ldots, f_n\} \subseteq A(h)$. Since each f_i is constant on the level sets of h, it follows that h is one-to-one, hence a homeomorphism of X onto a compact subset of $\mathbb{R}$. □

Example 4.5. Let X be the one-point compactification of a topological sum of ω_1 copies of $I = [0, 1]$ (here ω_1 denotes the first uncountable ordinal). Then X is not totally disconnected, and does not embed in $\mathbb{R}$, but $\mathrm{cr}(X) = 1$, so that X is an A-space.

Indeed $C(X) = \bigcup\{H(\beta)\colon\ \beta < \omega_1\}$, where

$$H(\beta) = \left\{f \in C(X)\colon\ f \text{ is constant on } \bigcup\{I_\alpha\colon\ \beta < \alpha < \omega_1\}\right\}.$$

Now $H(\beta)$ is isomorphic as a C^*-algebra to $C(X_\beta)$, where X_β is the one-point compactification of a topological sum of β copies of I. Each space X_β embeds in $\mathbb{R}$, so $\mathrm{cr}(X_\beta) = 1$. Now $f, g \in C(X)$ implies that $f, g \in H(\beta)$ for some $\beta < \omega_1$, and the result follows.

The preceding example and S^1 both enjoy the following property: every closed nowhere dense subset is totally disconnected. Corollary 6.3 will tell us that any compact space with this property is an A-space. However, the space

$$X = \{(x, \sin 1/x)\colon\ 0 < x \leqslant 1\} \cup \{(0, y)\colon\ -1 \leqslant y \leqslant 1\}$$

is a compact subset of $\mathbb{R}^2$ which fails this property, yet is an A-space, by Theorem 6.1.

The notion of continuous rank should be compared with the classical concept of analytic dimension, $\mathrm{ad}(X)$ (see [10, pp. 256–261] or [13, pp. 387–401]). A norm-closed subalgebra B of $C_b(X)$, containing 1, is said to be analytic if $f \in C_b(X)$ and $f^2 \in B$ implies that $f \in B$. The definition of $\mathrm{ad}(X)$ is then identical to that of $\mathrm{cr}(X)$, except that $A(H)$ is replaced by $B(H)$, the smallest analytic subalgebra of $C_b(X)$ which contains H. Clearly $A(H) \subseteq B(H)$, so that $\mathrm{ad}(X) \leqslant \mathrm{cr}(X)$.

If X is compact, then $A(H) = \{f \in C(X)\colon\ f$ is constant on each set of constancy of $H\}$, and $B(H) = \{f \in C(X)\colon\ f$ is constant on each connected component of each set of constancy of $H\}$. This difference can be extreme: consider X totally disconnected and $H = \{1\}$. For $X = S^1$, $C(X) = B(h)$, where $h(x, y) = x$, so $\mathrm{ad}(X) < \mathrm{cr}(X)$.

If X is normal, then a theorem of Katetov ([10, 16.35] or [13, 10.4.12]) asserts that $\mathrm{ad}(X) = \dim(X)$. The significance of the condition $\dim(X) \leqslant 1$ for $\{0, 1\}$-valued quasi-measures (and multiplicative quasi-linear functionals) will become clear in Section 5.

5. $\dim(X)$ and $\{0,1\}$-valued quasi-measures

When X is a compact Hausdorff space, Aarnes [4] has shown that the $\{0,1\}$-valued Borel quasi-measures (called by him "extremal") correspond to the quasi-linear functionals $\rho: C(X) \to \mathbb{R}$ such that $\rho(1) = 1$ and the restriction of ρ to each $A(h)$ is multiplicative. The set Y of these "simple quasi-states" is a compact Hausdorff space in the topology of pointwise convergence on members of $C(X)$, and contains X (more precisely, the set of evaluation functionals at points of X) as a closed subspace. The question of relating topological properties of X to those of the associated space Y appears to be a very interesting one, following the lead of [4].

The next result extends this correspondence to normal spaces.

Proposition 5.1. *Let X be a normal space, and let ρ be a quasi-linear functional on $C_b(X)$ with $\rho(1) = 1$. Then the following are equivalent:*
(a) *ρ is multiplicative on each $A(h) \subseteq C_b(X)$.*
(b) *ρ^β is multiplicative on each $A(h^\beta) \subseteq C(\beta X)$.*
(c) *ν is $\{0,1\}$-valued on the closed subsets of X.*
(d) *ν^β is $\{0,1\}$-valued on the closed subsets of βX.*

Proof. (a) $\Leftrightarrow$ (b): Since $(f \cdot g)^\beta = f^\beta \cdot g^\beta$, this is clear.
(b) $\Leftrightarrow$ (d): [4].
(d) $\Rightarrow$ (c): Let F_1 be a closed subset of X with $\nu(F_1) > 0$. Since

$$\nu(X) = \rho(1) = \rho^\beta(1^\beta) = \nu^\beta(\beta X) = 1,$$

given $\varepsilon > 0$, we can choose a closed subset F_2 of X with $F_1 \cap F_2 = \emptyset$ and $\nu(F_1) + \nu(F_2) > 1 - \varepsilon$, using Definition 2.1(iii). Let $K_i = \mathrm{cl}_{\beta X}(F_i)$, $i = 1, 2$; by normality, $K_1 \cap K_2 = \emptyset$. If $W \in \mathcal{Z}(\beta X)$ and $W \supseteq K_1$, then

$$\nu^\beta(W) \geqslant \nu(W \cap X) \geqslant \nu(F_1),$$

and so $\nu^\beta(K_1) = \inf\{\nu^\beta(W):\ W \supseteq K_1\} \geqslant \nu(F_1)$. Similarly, $\nu^\beta(K_2) \geqslant \nu(F_2)$. Since $\nu(F_1) > 0$, (d) implies that $\nu^\beta(K_1) = 1$, so $\nu^\beta(K_2) = \nu(F_2) = 0$, and $1 - \varepsilon < \nu(F_1) \leqslant \nu(X) = 1$. Thus (c) holds.

(c) $\Rightarrow$ (d): Suppose K_1 is a closed subset of βX with $\nu^\beta(K_1) < 1$. Since $\nu^\beta(\beta X) = \nu(X) = 1$, we can choose a closed subset K_2 of βX with $K_1 \cap K_2 = \emptyset$ and $\nu^\beta(K_2) > 0$, using Definition 2.1(iii). Choose $g \in C(\beta X)$ with $0 \leqslant g \leqslant 1$, $g|K_1 \equiv 1$, and $g|K_2 \equiv 0$. Let

$$T_1 = \{p \in \beta X:\ g(p) \geqslant 1/2\}.$$

Then $\nu(T_1 \cap X) \leqslant \nu^\beta(T_1) \leqslant \nu^\beta(\beta X \setminus K_2) < 1$, and so $\nu(T_1 \cap X) = 0$, by (c). Choose a closed subset F_2 of X with $(T_1 \cap X) \cap F_2 = \emptyset$ and $\nu(F_2) = 1$. Now $\mathrm{cl}_{\beta X}(T_1 \cap X) \cap \mathrm{cl}_{\beta X}(F_2) = \emptyset$, by normality, and

$$\mathrm{cl}_{\beta X}(T_1 \cap X) \supseteq \{p \in \beta X:\ g(p) > 1/2\} \supseteq \{p \in \beta X:\ g(p) \geqslant 3/4\} \supseteq K_1.$$

Then $T_2 = \{p \in \beta X\colon\ g(p) \geqslant 3/4\}$ is disjoint from $T_3 = \mathrm{cl}_{\beta X}(F_2)$, and $1 \geqslant \nu^\beta(T_2 \cup T_3) = \nu^\beta(T_2)+\nu^\beta(T_3) \geqslant \nu^\beta(K_1)+\nu^\beta(T_3) \geqslant \nu^\beta(K_1)+\nu(T_3\cap X) = \nu^\beta(K_1)+\nu(F_2) = \nu^\beta(K_1) + 1$, so that $\nu^\beta(K_1) = 0$, as desired. □

Definition 5.2. A normal space X is a *multiplicative A-space* (*MA-space*) if every multiplicative quasi-linear functional (in the sense of Proposition 5.1(a)) on $C_b(X)$ is linear.

In view of Theorem 3.3 and Proposition 5.1, it is equivalent to require that every $\{0,1\}$-valued quasi-measure ν on X correspond to a point mass δ_p at some point $p \in \beta X$. This means that $\nu(F) = 1$ if $p \in \mathrm{cl}_{\beta X}(F)$, and $\nu(F) = 0$ otherwise. It is easy to see (as in Section 4) that X is an MA-space if and only if βX is an MA-space, and that the property is preserved by closed subspaces.

The next result (equivalent to one obtained in [12]) provides a purely topological way of looking at $\{0,1\}$-valued quasi-measures. The proof is left to the reader.

Proposition 5.3. *Let X be a normal space.*

(a) *If ν is a $\{0,1\}$-valued Borel quasi-measure on X, then $\mathcal{M} = \{F \in \mathcal{F}\colon\ \nu(F) = 1\}$ is a nonempty collection of nonempty closed subsets of X satisfying:*

(i) $F_1 \in \mathcal{M}, F_1 \subseteq F_2 \Rightarrow F_2 \in \mathcal{M}$.

(ii) $F_1, F_2 \in \mathcal{M} \Rightarrow F_1 \cap F_2 \neq \emptyset$.

(iii) $F_1 \cup F_2 \in \mathcal{M}, F_1 \cap F_2 = \emptyset \Rightarrow F_1 \in \mathcal{M}$ *or* $F_2 \in \mathcal{M}$.

(iv) $F_1 \notin \mathcal{M} \Rightarrow \exists F_2 \in \mathcal{M}, F_1 \cap F_2 = \emptyset$.

(b) *If $\mathcal{M}$ is a nonempty collection of nonempty closed subsets of X satisfying* (i)–(iv), *then $\nu : \mathcal{F} \to \mathbb{R}$ defined by $\nu(F) = 1$ if $F \in \mathcal{M}, \nu(F) = 0$ if $F \notin \mathcal{M}$ is a quasi-measure on X.*

We call a collection $\mathcal{M}$ of this type a *maximal quasifilter* of closed subsets of X. Any ultrafilter of closed subsets of X is a maximal quasifilter; conversely, a maximal quasifilter $\mathcal{M}$ is an ultrafilter of closed sets precisely when it satisfies the stronger property (ii′): $F_1, F_2 \in \mathcal{M} \to F_1 \cap F_2 \in \mathcal{M}$. This is exactly what is required for ν to correspond to a point mass δ_p at some point $p \in \beta X$, and then $\mathcal{M} = \{F \in \mathcal{F}\colon\ p \in \mathrm{cl}_{\beta X}(F)\}$.

In the example from [3, Section 6], it is easy to find $F_1, F_2 \in \mathcal{M}$ such that $F_1 \cap F_2 \notin \mathcal{M}$ (and also $F_1, F_2, F_3 \in \mathcal{M}$ such that $F_1 \cap F_2 \cap F_3 = \emptyset$).

If a Zorn's Lemma argument is applied to a collection of closed sets satisfying (i), (ii), and (iii), then it is not clear that any "maximal extension" thus obtained will satisfy (iv). Consequently, it seems to be necessary to construct maximal quasifilters which are not ultrafilters by a bare-hands approach. This is the remarkable achievement of [3], systematized in [5,12]. It would be interesting to develop a comprehensive theory of maximal quasifilters along the lines of [10].

We conclude this section by relating $\{0,1\}$-valued quasi-measures to the Lebesgue covering dimension of the underlying space in a natural way. Note that $\dim(X) = \dim(\beta X)$, and $\dim(F) \leqslant \dim(X)$ for each closed subset F of X. The question as to whether a normal space X such that $\dim(X) \leqslant 1$ must be an A-space seems to be open, and is perhaps related to the concept of "representable" quasi-measures [4].

Theorem 5.4. *If X is a normal space and* $\dim(X) \leqslant 1$, *then X is an MA-space.*

Proof. The statement that $\dim(X) \leqslant 1$ means precisely that if $X = U_1 \cup U_2 \cup U_3$, each U_i open, then there exist open sets V_1, V_2, V_3 with $V_1 \cup V_2 \cup V_3 = X$, $V_1 \cap V_2 \cap V_3 = \emptyset$, and $V_i \subseteq U_i$ for each i [13, p. 111]. Equivalently, we have:

($*$) If F_1, F_2, F_3 are closed subsets of X with $F_1 \cap F_2 \cap F_3 = \emptyset$, then there exist closed sets T_1, T_2, T_3 with $T_1 \cap T_2 \cap T_3 = \emptyset$, $T_1 \cup T_2 \cup T_3 = X$, and $T_i \supseteq F_i$ for each i.

Now let $\mathcal{M}$ be a maximal quasifilter of closed subsets of X. Then we have:

($**$) If $H_1, H_2 \in \mathcal{M}$ and $H_1 \cup H_2 = X$, then $H_1 \cap H_2 \in \mathcal{M}$.

Indeed in this case, we can express X as the disjoint union of $X \setminus H_1, X \setminus H_2$, and $H_1 \cap H_2$. But $\nu(X \setminus H_1) = \nu(X \setminus H_2) = 0$, so $\nu(H_1 \cap H_2) = 1$.

Suppose $\mathcal{M}$ is not an ultrafilter, and choose $F_1, F_2 \in \mathcal{M}$ (so that $F_1 \cap F_2 \neq \emptyset$, by Proposition 5.3(ii)) with $F_1 \cap F_2 \notin \mathcal{M}$. By (iv), there is an $F_3 \in \mathcal{M}$ with $F_1 \cap F_2 \cap F_3 = \emptyset$.

Choose T_1, T_2, T_3 as in ($*$). Since $T_i \supseteq F_i$ for each i, $T_1, T_2, T_3 \in \mathcal{M}$, by (i). Now $T_1 \cup T_2 \in \mathcal{M}$, and $(T_1 \cup T_2) \cup T_3 = X$, so $(T_1 \cup T_2) \cap T_3 = (T_1 \cap T_3) \cup (T_2 \cap T_3) \in \mathcal{M}$, by ($**$). Since $(T_1 \cap T_3) \cap (T_2 \cap T_3) = \emptyset$, either $T_1 \cap T_3 \in \mathcal{M}$ or $T_2 \cap T_3 \in \mathcal{M}$, by (iii). To be definite, say that $T_1 \cap T_3 \in \mathcal{M}$, $T_2 \cap T_3 \notin \mathcal{M}$.

Now $T_1 \cup T_3 \in \mathcal{M}$ and $(T_1 \cup T_3) \cup T_2 = X$, so $(T_1 \cup T_3) \cap T_2 = (T_1 \cap T_2) \cup (T_2 \cap T_3) \in \mathcal{M}$. Since $T_2 \cap T_3 \notin \mathcal{M}$, (iii) implies that $T_1 \cap T_2 \in \mathcal{M}$. But then $(T_1 \cap T_3) \cap (T_1 \cap T_2) \neq \emptyset$ by (ii), a contradiction. Hence $\mathcal{M}$ is an ultrafilter, and the result follows from the remarks after Definition 5.2 and Proposition 5.3. □

6. Quasi-measures and Ind(X)

If X is a normal space, then the large inductive dimension $\mathrm{Ind}(X) = 0$ if and only if for each pair F_1, F_2 of disjoint closed subsets of X, there is an open set U with $F_1 \subseteq U \subseteq X \setminus F_2$ and $\partial(U) = \emptyset$ [13, p. 155]. This amounts to saying that disjoint closed sets can be separated by disjoint clopen sets, and is equivalent to the statement that $\dim(X) = 0$. Also, $\mathrm{Ind}(X) \leqslant 1$ if and only if for each pair F_1, F_2 of disjoint closed subsets of X, there is an open set U with $F_1 \subseteq U \subseteq X \setminus F_2$ and $\mathrm{Ind}(\partial(U)) \leqslant 0$ (by convention, $\mathrm{Ind}(\emptyset) = -1$). Applying the fact that X is normal, we may assume that $\mathrm{cl}(U)$, and hence $\partial(U)$, is a subset of $X \setminus F_2$.

We have $\mathrm{Ind}(X) = \mathrm{Ind}(\beta X)$, and $\mathrm{Ind}(F) \leqslant \mathrm{Ind}(X)$ for each closed subset F of X. Also, $\dim(X) \leqslant \mathrm{Ind}(X)$, and equality holds if X is metrizable [13, p. 181].

The definition of Ind(X) fits quite naturally into the proof of Theorem 6.1, the main result of this paper.

Theorem 6.1. *If X is a normal space and* $\mathrm{Ind}(X) \leqslant 1$, *then X is an A-space.*

The key to the proof is the following "excision lemma", which may be of independent interest.

Lemma 6.2. *Let ν be a Borel quasi-measure on a normal space X. Then there is a finitely additive, closed regular measure η on $Bo^*(X)$ such that:*
(i) $0 \leqslant \eta(F) \leqslant \nu(F)$ *for all* $F \in \mathcal{F}$.
(ii) $\eta(D) = \nu(D)$ *for each* $D \in \mathcal{F}$ *such that* $\operatorname{Ind}(D) = 0$.
(iii) $\gamma = \nu - \eta$ *is a Borel quasi-measure on* X *such that* $\gamma(D) = 0$ *for each* $D \in \mathcal{F}$ *such that* $\operatorname{Ind}(D) = 0$.

Proof. Note that (iii) follows from (i), (ii), and Proposition 3.4. Let

$$\mathcal{D} = \{D \in \mathcal{F}\colon \operatorname{Ind}(D) = 0\};$$

this collection is preserved by finite unions [13, pp. 125, 157]. Fix $D \in \mathcal{D}$, and let $\mathcal{B}$ denote the Boolean algebra of subsets of D which are clopen in the relative topology of D. Each member of $\mathcal{B}$ is closed in X, and $\lambda = \nu|\mathcal{B}$ is finitely additive. Since $\operatorname{Ind}(D) = 0$, $\mathcal{B}$ is a base for the relative topology of D.

Since X is normal and D is closed in X, $\beta D = \mathrm{cl}_{\beta X}(D)$ and $\operatorname{Ind}(\beta D) = \operatorname{Ind}(D) = 0$. If $\mathcal{B}'$ is the Boolean algebra of clopen subsets of βD, then $\mathcal{B}'$ is isomorphic (as a Boolean algebra) to $\mathcal{B}$ via $C' \to C' \cap D$ (and, inversely, $C \to \mathrm{cl}_{\beta D}(C)$). The set function $\lambda' : \mathcal{B}' \to \mathbb{R}$ defined by $\lambda'(C') = \lambda(C' \cap D)$ is finitely additive, and (by standard Stone representation theory) extends uniquely to a countably additive, closed regular measure α' on $Bo(\beta D)$. This in turn corresponds to a unique finitely additive, closed regular measure α on $Bo^*(D)$ such that

$$\int_D f \,\mathrm{d}\alpha = \int_{\beta D} f^\beta \,\mathrm{d}\alpha'$$

for each $f \in C_b(D)$ [14, Section 5 and Theorem 9.8]. We have $\alpha|\mathcal{B} = \lambda = \nu|\mathcal{B}$, for if $C \in \mathcal{B}$ and $C' = \mathrm{cl}_{\beta D}C$, then

$$\alpha(C) = \int_D \chi_C \,\mathrm{d}\alpha = \int_{\beta D} \chi_{C'} \,\mathrm{d}\alpha' = \alpha'(C') = \lambda'(C') = \lambda(C).$$

Using the injection $i : D \to X$, we may think of α as a finitely additive, closed regular measure on $Bo^*(X)$.

We claim that $\alpha(F) = \nu(F)$ for each closed subset F of D. To see this, let $\varepsilon > 0$, and choose a relatively open subset U of D with $F \subseteq U$ and $\alpha(F) \leqslant \alpha(U) < \alpha(F) + \varepsilon$ (note that $\nu(U)$ need not be defined). Since $D \in \mathcal{D}$, we can choose $C \in \mathcal{B}$ with $F \subseteq C \subseteq U$. Then $\nu(F) \leqslant \nu(C) = \alpha(C) \leqslant \alpha(U) < \alpha(F) + \varepsilon$, and so $\nu(F) \leqslant \alpha(F)$. Conversely, given $\varepsilon > 0$, there is an open subset V of X with $F \subseteq V$ and $\nu(F) \leqslant \nu(V) < \nu(F) + \varepsilon$. Since $D \in \mathcal{D}$, we can choose $C \in \mathcal{B}$ with $F \subseteq C \subseteq V \cap D$. Then $\alpha(F) \leqslant \alpha(C) = \nu(C) \leqslant \nu(V) < \nu(F) + \varepsilon$, and so $\alpha(F) \leqslant \nu(F)$.

Note that if $\operatorname{Ind}(X) = 0$, then this much of the proof already shows that X is an A-space, using Theorem 3.3(d). This is the measure-theoretic analogue of the functional analysis proof for compact zero-dimensional spaces given in [6].

Continuing with the general case, let $c = \sup\{\nu(D)\colon D \in \mathcal{D}\}$. Choose a sequence $\{D_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ with $\lim \nu(D_n) = c$. Since $\mathcal{D}$ is preserved by finite unions, we may assume that $D_1 \subseteq D_2 \subseteq \cdots$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of finitely additive, closed regular measures on $Bo^*(X)$, formed as above, so that α_n agrees with ν on all closed subsets of D_n. If F is any closed subset of X, then $\alpha_n(F) = \alpha_n(F \cap D_n) = \nu(F \cap D_n) \leqslant \nu(F \cap D_{n+1}) = \alpha_{n+1}(F)$, and it follows from closed regularity that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$ in the complete lattice $M(X)$ of finitely additive, closed regular measures on $Bo^*(X)$ (cf. [14, p. 114]). Let $\eta = \sup \alpha_n$ in $M(X)$. Then $0 \leqslant \eta(F) = \sup \alpha_n(F) = \sup \nu(F \cap D_n) \leqslant \nu(F)$ for all $F \in \mathcal{F}$. This establishes part (i) of the lemma.

Now let D be an arbitrary but fixed member of $\mathcal{D}$, and suppose that $\nu(D) - \eta(D) = \delta > 0$. Choose n such that $\nu(D_n) > c - \delta$. Let $E = D \cup D_n$, again a member of $\mathcal{D}$, and extend $\nu|\,\mathrm{clopen}(E)$ to a finitely additive, closed regular measure α on $Bo^*(E)$, as before. Then $\alpha(F) = \nu(F)$ for each closed subset F of E, by the claim above.

We have $\alpha(D \setminus D_n) = \alpha(D) - \alpha(D \cap D_n) = \nu(D) - \nu(D \cap D_n) = \nu(D) - \alpha_n(D) \geqslant \nu(D) - \eta(D) = \delta$. Thus $\nu(E) = \alpha(E) = \alpha(D_n) + \alpha(D \setminus D_n) = \nu(D_n) + \alpha(D \setminus D_n) > (c - \delta) + \delta = c$, contradicting the definition of c. Thus $\nu(D) = \eta(D)$ for all $D \in \mathcal{D}$, completing the proof of (ii). □

Corollary 6.3. *If X is normal, and every nonempty closed nowhere dense subset D satisfies* $\mathrm{Ind}(D) = 0$, *then X is an A-space.*

Proof. This is immediate from Corollary 3.2, Theorem 3.3, and Lemma 6.2. □

Since the hypothesis of Corollary 6.3 implies that $\mathrm{Ind}(\partial(U)) \leqslant 0$ for every open subset U of X, this result is a special case of Theorem 6.1. The space $\mathbb{R}$ has the stated property.

At this point, we apply the lemma to complete the proof of Theorem 6.1.

Proof of Theorem 6.1. Let ν be a Borel quasi-measure on X. By Theorem 3.3, it suffices to show that if A_1, A_2 are closed subsets of X, then for any $\varepsilon > 0$, $\nu(A_1 \cup A_2) \leqslant \nu(A_1) + \nu(A_2) + \varepsilon$. By Lemma 6.2, we may assume that $\nu(D) = 0$ for every closed subset D such that $\mathrm{Ind}(D) = 0$.

Choose closed sets B_1, B_2 with $B_i \cap A_i = \emptyset$ and $\nu(A_i) + \nu(B_i) > \nu(X) - \varepsilon/2$, $i = 1, 2$. Since $\mathrm{Ind}(X) \leqslant 1$, we can choose open sets U_1, U_2 with $A_i \subseteq U_i \subseteq \mathrm{cl}(U_i) \subseteq X \setminus B_i$ and $\mathrm{Ind}(\partial(U_i)) \leqslant 0$, $i = 1, 2$. Then $\mathrm{Ind}(\partial(U_1) \cup \partial(U_2)) \leqslant 0$, and so $\nu(\partial(U_1) \cup \partial(U_2)) = 0$.

Now $\nu(A_1 \cup A_2) \leqslant \nu(\mathrm{cl}(U_1) \cup \mathrm{cl}(U_2)) = \nu(U_1 \cup \mathrm{int}(U_2 \setminus U_1) \cup D)$, where $D = \mathrm{cl}(U_1) \cup \mathrm{cl}(U_2) \setminus (U_1 \cup \mathrm{int}(U_2 \setminus U_1))$ is closed, and the three sets are pairwise disjoint. It now suffices to show that $D \subseteq \partial(U_1) \cup \partial(U_2)$, since then $\nu(D) = 0$, and $\nu(A_1 \cup A_2) \leqslant \nu(U_1) + \nu(\mathrm{int}(U_2 \setminus U_1)) \leqslant \nu(U_1) + \nu(U_2) = \nu(A_1) + \nu(U_1 \setminus A_1) + \nu(A_2) + \nu(U_2 \setminus A_2) \leqslant \nu(A_1) + \nu(X \setminus (A_1 \cup B_1)) + \nu(A_2) + \nu(X \setminus (A_2 \cup B_2)) < \nu(A_1) + \nu(A_2) + \varepsilon$.

If $x \in D$, then $x \in \mathrm{cl}(U_1) \cup \mathrm{cl}(U_2)$. If $x \in \mathrm{cl}(U_1)$, then $x \in \mathrm{cl}(U_1) \setminus U_1 = \partial(U_1)$. If $x \in \mathrm{cl}(U_2)$, then either $x \in \partial(U_2)$ or $x \in U_2$. In the latter case, $x \in U_2 \setminus (U_1 \cup \mathrm{int}(U_2 \setminus U_1))$, so $x \in U_2 \setminus U_1$, but $x \notin \mathrm{int}(U_2 \setminus U_1)$. Hence $x \notin U_2 \setminus \mathrm{cl}(U_1)$. However, $x \in U_2$, so $x \in \mathrm{cl}(U_1)$. Thus $x \in \mathrm{cl}(U_1) \setminus U_1 = \partial(U_1)$, and so $D \subseteq \partial(U_1) \cup \partial(U_2)$. This completes the proof. □

Example 6.4. $X = \{(x,y) \in [0,1] \times [0,1]\colon\ x \in Q \text{ or } y \in Q\}$, the "rational square" in the plane.

The space X is a separable metric space, and a countable union of copies of $[0,1]$, so $\mathrm{Ind}(X) = \dim(X) = 1$ [13, p. 125]. Thus X is an A-space, by Theorem 6.1, and so the example in [3, Section 6] cannot be "restricted" to the dense subset X of $[0,1] \times [0,1]$. Indeed by using horizontal and vertical lines determined by irrational coordinates, it is easy to construct two disjoint rectangular "curves" in X, each of which would have measure 1 under the supposed restriction.

The following diagram summarizes the relation of Borel quasi-measures on normal spaces to various notions of dimension, as developed in this article. At any node, X can be replaced by βX. Let $\mathcal{N}$ denote the collection of closed, nowhere dense subsets of X. Recall that $\dim(X) = \mathrm{ad}(X) \leqslant \mathrm{cr}(X)$.

Diagram 6.5. *Normal spaces*

$$\begin{array}{ccccccc}
\mathrm{cr}(X) \leqslant 1 & \longrightarrow & & \longrightarrow & A\text{-space} & \longrightarrow & MA\text{-space} \\
\uparrow & & & & \uparrow & & \uparrow \\
\mathrm{Ind}(X) = 0 & \longrightarrow & \mathrm{Ind}(D) = 0\ \forall D \in \mathcal{N} & \longrightarrow & \mathrm{Ind}(X) \leqslant 1 & \longrightarrow & \dim(X) \leqslant 1
\end{array}$$

The space $X = S^1$ is a compact A-space for which all closed nowhere dense subsets are zero-dimensional, but $\mathrm{cr}(X) = 2$. The example given in Section 7 is a compact connected A-space S with $\dim(S) = 1$, but $\mathrm{cr}(S)$ and $\mathrm{Ind}(S)$ are both greater than 1.

If we restrict attention to compact metric spaces X, then $\mathrm{Ind}(X) = \dim(X)$; and if $\mathrm{cr}(X) \leqslant 1$, then X embeds in $\mathbb{R}^3$ (see [9, p. 311]). Using Proposition 4.4, the diagram above can now be linearized.

Diagram 6.6. *Compact metric spaces*

$$\begin{aligned}
&\mathrm{Ind}(X) = 0 \to \mathrm{cr}(X) \leqslant 1 \to \mathrm{Ind}(D) = 0\ \forall D \in \mathcal{N} \\
&\qquad \to \mathrm{Ind}(X) \leqslant 1 \to A\text{-space} \to MA\text{-space}
\end{aligned}$$

Here it is not known if the last two implications can be reversed.

7. An A-space S with $\dim(S) = 1$, $\mathrm{Ind}(S) = 2$

The construction of the space S is given in [13, pp. 162–166]. The space S is a union of two closed subsets S_1 and S_2 which satisfy $\dim(S_i) = \mathrm{Ind}(S_i) = 1$, $i = 1, 2$, and $S_1 \cap S_2$ is homeomorphic to $[0,1]$. The open sets $\mathrm{int}(S_1) = S_1 \setminus S_2$ and $\mathrm{int}(S_2) = S_2 \setminus S_1$ are each homeomorphic to $L_0 \times C$, where $L_0 = \{\xi = (\alpha, r)\colon\ 0 \leqslant \alpha < \omega_1, 0 \leqslant r < 1\}$ is the "long line" with the endpoint ω_1 omitted, and C is the Cantor set. Let $D = \{y \in C\colon\ y$ is an endpoint of an interval in $[0,1] \setminus C\}$. We write $y_1 \sim y_2$ if y_1, y_2 are the two endpoints of an interval in $[0,1] \setminus C$.

If $f \in C(S)$, then $f|\operatorname{int}(S_1)$ and $f|\operatorname{int}(S_2)$ are eventually constant on each "horizontal line" determined by $y \in C$. Since D is countable and dense in C, it follows that if $f \in C(S)$, then:

($*$) there is $\xi_0 \in L_0$ such that if $y \in C$ and $\xi_0 \leqslant \xi_1 < \xi_2 < \omega_1$ in L_0, then $f(1, \xi_1, y) = f(1, \xi_2, y)$ and $f(2, \xi_1, y) = f(2, \xi_2, y)$, where 1 and 2 denote points in $\operatorname{int}(S_1)$ and $\operatorname{int}(S_2)$, respectively.

Moreover, the construction of S given in [13] shows that for each $y_1, y_2 \in D$ with $y_1 \sim y_2$, there is a unique $z_1 \in C \setminus D$ such that as $\xi \to \omega_1$,

$$\lim(1, \xi, y_1) = \lim(1, \xi, y_2) = \lim(2, \xi, z_1) = (\omega_1, t_1)$$

is a point of $M_1 \subseteq S_1 \cap S_2$; and there is a unique $z_2 \in C \setminus D$ such that as $\xi \to \omega_1$,

$$\lim(2, \xi, y_1) = \lim(2, \xi, y_2) = \lim(1, \xi, z_2) = (\omega_1, t_2)$$

is a point of $M_2 \subseteq S_1 \cap S_2$. Here M_1 and M_2 are countable dense subsets of $\{(\omega_1, t)\colon 0 < t < 1\}$, and $M_1 \cap M_2 = \emptyset$.

Thus for $f \in C(S)$ and ξ_0 as described in ($*$), we have:

($**$) $f(1, \xi_0, y_1) = f(1, \xi_0, y_2) = f(2, \xi_0, z_1) = f(\omega_1, t_1)$; and $f(2, \xi_0, y_1) = f(2, \xi_0, y_2) = f(1, \xi_0, z_2) = f(\omega_1, t_2)$.

Now let $H(\xi_0) = \{f \in C(S)\colon f$ satisfies ($*$) and ($**$) for $\xi_0\}$, where $0 < \xi_0 < \omega_1$ in L_0. By the preceding remarks,

$$C(S) = \bigcup \{H(\xi_0)\colon 0 < \xi_0 < \omega_1\}.$$

Each $H(\xi_0)$ is isomorphic as a C^*-algebra to $C(S(\xi_0))$, where $S(\xi_0)$ is constructed in the same manner as S, but using a "short line" $[0, \xi_0]$ instead of $[0, \omega_1]$. Explicitly, $S(\xi_0)$ is obtained by forming two Hausdorff quotients of the compact metric space $[0, \xi_0] \times C$, and then taking a continuous (Hausdorff) image of their disjoint union. As a result, $S(\xi_0)$ is again a compact metric space, and $\operatorname{Ind}(S(\xi_0)) = \dim(S(\xi_0)) = 1$. Thus $S(\xi_0)$ is an A-space, by Theorem 6.1.

Since each $S(\xi_0)$ contains a closed nowhere dense subset which is homeomorphic to $[0, 1]$, $\operatorname{cr}(S(\xi_0)) > 1$, using Diagram 6.6. Then also $\operatorname{cr}(S) > 1$.

In order to show that S is an A-space, let $\rho\colon C(S) \to \mathbb{R}$ be quasi-linear. Then for any $\xi < \omega_1$, the restriction of ρ to $H(\xi)$ corresponds to a quasi-linear functional ρ_ξ on $C(S(\xi))$, which is then linear, since $S(\xi)$ is an A-space. Since $\xi_1 < \xi_2$ implies that $H(\xi_1) \subseteq H(\xi_2)$, it is clear that if $f, g \in C(S)$, then there is $\xi_0 < \omega_1$ such that $f, g \in H(\xi_0)$. Thus (with a slight abuse of notation) $\rho(f + g) = \rho_{\xi_0}(f + g) = \rho_{\xi_0}(f) + \rho_{\xi_0}(g) = \rho(f) + \rho(g)$. Hence S is an A-space.

8. Extensions to the completely regular setting

Although we have chosen to work with Borel quasi-measures on normal spaces in this paper, many of the results obtained here have valid analogues for Baire quasi-measures on completely regular spaces, often with very similar proofs. For example, the analogue of Proposition 3.1 holds using zero sets in (a), cozero sets in (b), and $Ba^*(X)$ in (c).

The fact that disjoint zero sets are contained in disjoint cozero sets (as in [10, 1.15]) is needed for (a) $\Rightarrow$ (b). The analogue of Theorem 3.3 also holds, using zero sets in (c) and $Ba^*(X)$ in (d), and Proposition 3.4 is valid for Baire quasi-measures.

The definitions of A-space (4.1) and MA-space (5.2) still make sense for completely regular X, and it is still true that X has either property if and only if βX does. Similarly, the continuous rank $\mathrm{cr}(X)$ can be defined for completely regular spaces, and $\mathrm{cr}(X) = \mathrm{cr}(\beta X)$. The result of Proposition 5.1 is valid for $\{0,1\}$-valued Baire measures, and Proposition 5.3 is easily modified so as to equate them with maximal quasifilters of zero sets of X. Theorem 5.4 is true for completely regular spaces, using the modified dimension $\partial\mathrm{im}(X) = \dim(\beta X)$ (see [10, Chapter 16] or [13, p. 368]). Similarly, Theorem 6.1 holds for completely regular spaces under the hypothesis $\mathrm{Ind}(\beta X) \leqslant 1$.

It would be desirable to have an analogue of Proposition 2.3 in the completely regular setting; however, neither direction in the proof seems to allow an obvious modification. Note that by a result of Bachman and Sultan [7] (see also [14, Theorem 9.8]), every finitely additive, zero set regular measure on $Ba^*(X)$ extends to a finitely additive, closed regular measure on $Bo^*(X)$. In some sense, what is needed is a "quasi-linear" version of the Hahn–Banach Theorem, but none seems to be presently available.

We conclude by noting that the original interest in quasi-linear functionals arose from questions about noncommutative C^*-algebras (stemming, in turn, from considerations in the mathematical foundations of quantum mechanics [1,2]). Do the ideas raised in this paper have any application to the noncommutative setting?

Note added in proof

Dmitri Shakhmatov has shown that Theorem 6.1 remains true if the condition $\mathrm{Ind}(X) \leqslant 1$ is replaced by the weaker condition $\dim(X) \leqslant 1$.

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